Composition-Diamond Lemma for Modules*

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Abstract: In this paper we give some relationships among the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" left modules (free modules over a free algebra). We give the Chibrikov's Composition-Diamond lemma for modules and show that Kang-Lee's Composition-Diamond lemma follows from this lemma. As applications, we also deal with highest weight module over the Lie algebra sl_2, Verma module over a Kac-Moody algebra, Verma module over Lie algebra of coefficients of a free conformal algebra and the universal enveloping module for a Sabinin algebra.

Key words: Gröbner-Shirshov basis; module; Lie algebra; Kac-Moody algebra; conformal algebras; Sabinin algebra.

AMS Mathematics Subject Classification(2000): 16S15, 13P10, 17A01, 17B67, 16D10

1 Introduction

Composition-Diamond lemma for modules was first formulated and proved by S.-J. Kang and K.-H. Lee in [15] and [19]. According to their approach, a Gröbner-Shirshov basis of a cyclic module M over an algebra A is a pair (S,T), where S is the set of the defining relations of A, $A=k\langle X|S\rangle$ and T is the defining relations for the A-module ${}_AM=mod_A\langle e|T\rangle$. Then Kang-Lee's Lemma says that (S,T) is a Gröbner-Shirshov pair for A-module ${}_AM=mod_A\langle e|T\rangle$ if S is a Gröbner-Shirshov basis of A and T is closed under the right-justified composition with respect to S, and for $f\in S,\ g\in T$, such that $(f,g)_w$ is defined and $(f,g)_w\equiv 0\ mod(S,T;w)$.

They gave applications of this lemma for irreducible modules over $sl_n(k)$ [16], Specht modules over Hecke algebras and Ariki-Koike algebras in [17] and [18]. Some years later, E. S. Chibrikov [11] suggested a new Composition-Diamond lemma for modules that treat

*Supported by the NNSF of China (No.10771077) and the NSF of Guangdong Province (No.06025062).

any module as a factor module of "double-free" module, a free module $mod_{k\langle X\rangle}\langle Y\rangle$ over a free algebra $k\langle X\rangle$. In this approach any A-module ${}_AM$ is presented in the form

$${}_AM = mod_{k\langle X\rangle}\langle Y|SX^*Y,\ T\rangle,$$

where $A = k\langle X|S\rangle,\ {}_AM = mod_A\langle Y|T\rangle,\ X^*$ is the free monoid generated by X.

The aim of this paper is to give some relationships among the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" left modules. Also we give some applications of Composition-Diamond lemma for "double-free" modules.

The paper is organized as follows. In §2, we are dealing with Gröbner-Shirshov bases and Composition-Diamond lemma for left ideals of a free algebra. Actually, it is a special case of cyclic "double-free" modules. In §3, we give some relationships among the Gröbner-Shirshov bases in free associative algebras, free left modules and "double-free" modules. We give a proof of Chibrikov's Composition-Diamond lemma and formulate Kang-Lee's Composition-Diamond lemma. Then, we show that the latter follows from the former. As applications, in §4, §5, §6 and §7, we are dealing with highest weight module over the Lie algebra sl_2, Verma module over a Kac-Moody algebra, Verma module over Lie algebra of coefficients of a free conformal algebra and the universal enveloping module for a Sabinin algebra, respectively.

Let k be a field, X a set, X^* the free monoid of associative words on X, and $k\langle X\rangle$ the free associative algebra over X and k. For a word $w \in X^*$, we denote the length of w by $deg(w)$. Suppose that " $<$ " is a well order on X^*. For any polynomial f, let $\overline{f}$ be the leading term of f. If the coefficient of $\overline{f}$ is 1, then we call this polynomial is monic.

The following lemma will be used in §4, §5 and §6.

Lemma 1.1 *([26], cf. [9], [10], [20], [3]) Let $Lie(X)$ be a free Lie algebra over a set X and a field k. Let $S \subset Lie(X)$ be a nonempty set of monic Lie polynomials. Then, with a deg-lex order on X^*, S is a Gröbner-Shirshov basis in $Lie(X)$ if and only if $S^{(-)}$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ where $S^{(-)}$ is just S but substitute all $[xy]$ by $xy - yx$.*

2 Composition-Diamond lemma for left ideals of a free algebra

Let X be a set and " $<$ " a well order on X^*. Let $S \subset k\langle X\rangle$ with each $s \in S$ monic. Then $k\langle X\rangle S$ is the left ideal of $k\langle X\rangle$ generated by S. For left ideal $k\langle X\rangle S$, we define the compositions in S as follows.

Definition 2.1 *For any $f,\ g \in S$, if $w = \overline{f} = a\overline{g}$ for some $a \in X^*$, then the composition of f and g is defined to be $(f,g)_w = f - ag$. Transformation $f \to f - ag$ is called the elimination of the leading word (ELW) of g in f.*

If $(f,g)_w = \sum \alpha_i a_i s_i$, where $\alpha_i \in k,\ a_i \in X^*,\ s_i \in S$ and $a_i\overline{s}_i < w$, then we say this composition is trivial $mod(S,w)$ and denote by $(f,g)_w \equiv 0 \ \ mod(S,w)$.

Definition 2.2 *Let $S \subset k\langle X\rangle$ with each $s \in S$ monic. Then we call S a Gröbner-Shirshov basis of left ideal $k\langle X\rangle S$ if all the compositions are trivial modulo S. S is called the minimal Gröbner-Shirshov basis of $k\langle X\rangle S$, if there are no compositions of polynomials in S, i.e. $\overline{f} \neq a\overline{g}$ for any $f,\ g \in S,\ f \neq g$.*

A well order " $<$ " on X^* is left compatible if for any $u, v \in X^*$,

$$u > v \Rightarrow wu > wv, \ \textit{for all} \ w \in X^*.$$

That " $<$ " is right compatible is similarly defined. " $<$ " is monomial if it is both left and right compatible.

Now we can formulate the Composition-Diamond lemma for left ideals of a free algebra.

Lemma 2.3 *(Composition-Diamond lemma for left ideals of $k\langle X\rangle$) Let $S \subset k\langle X\rangle$ with each $s \in S$ monic and " $<$ " a left compatible order on X^*. Then the following statements are equivalent:*

(1) *S is a Gröbner-Shirshov basis of left ideal $k\langle X\rangle S$.*

(2) *If $0 \neq f \in k\langle X\rangle S$, then $\overline{f} = a\overline{s}$ for some $a \in X^*,\ s \in S$.*

(2') *If $0 \neq f \in k\langle X\rangle S$, then $f = \sum \alpha_i a_i s_i$ with $a_1\overline{s}_1 > a_2\overline{s}_2 > \cdots$, where each $a_i \in X^*,\ s_i \in S$.*

(3) *$Red(S) = \{w \in X^* | w \neq a\overline{s},\ a \in X^*,\ s \in S\}$ is a k-linear basis for the factor $k\langle X\rangle$-module ${}_{k\langle X\rangle}k\langle X\rangle/k\langle X\rangle S$.*

Lemma 2.3 is a special case of Lemma 3.2 (see the next section).

Assume that S is a Gröbner-Shirshov basis for the left ideal $k\langle X\rangle S$ with no compositions at all between different elements of S. It means that S is a minimal Gröbner-Shirshov basis of the left ideal $k\langle X\rangle S$. Then, $k\langle X\rangle S$ is a free $k\langle X\rangle$-module with the basis S.

Now, we cite Kang-Lee's Composition-Diamond lemma.

Let $S,\ T \subset k\langle X\rangle,\ A = k\langle X|S\rangle,\ {}_AM = {}_A A/A(T + Id(S))$ a left A-module and $f, g \in k\langle X\rangle$. In Kang-Lee's paper [15], the composition of f and g is defined as follows.

Definition 2.4 *([15],[19])*

(a) *If there exist $a,\ b \in X^*$ such that $w = \overline{f}a = b\overline{g}$, with $deg(\overline{f}) > deg(b)$, then the composition of intersection is defined to be $(f, g)_w = fa - bg$.*

(b) *If there exist $a,\ b \in X^*$ such that $w = a\overline{f}b = \overline{g}$, then the composition of inclusion is defined to be $(f, g)_w = afb - g$.*

(c) *A composition $(f, g)_w$ is called right-justified if $w = \overline{f} = a\overline{g}$ for some $a \in X^*$.*

If $f - g = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i,\ \beta_j \in k,\ a_i,\ b_i,\ c_j \in X^*,\ s_i \in S,\ t_j \in T$ with $a_i\overline{s}_i b_i < w$ and $c_j\overline{t}_j < w$ for each i and j, then $f - g$ is called trivial with respect to S and T and denote it by $f \equiv g\ mod(S, T; w)$.

When $T = \varnothing$, we simply write $f \equiv g\ mod(S, w)$. If for any $f,\ g \in S,\ (f, g)_w$ is defined and $f \equiv g\ mod(S, w)$, then we say S is closed under the composition. Note that, if this is the case, S is called a Gröbner-Shirshov basis in $k\langle X\rangle$ which is firstly introduced by Shirshov [26] (see also [1], [2]).

Remark. If a subset S of $k\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^c in $k\langle X\rangle$. Such a process is called the Shirshov algorithm.

Definition 2.5 *([15],[19]) A pair (S,T) of subsets of monic elements of $k\langle X\rangle$ is called a Gröbner-Shirshov pair if S is closed under the composition, T is closed under the right-justified composition with respect to S and T, and for any $f \in S,\ g \in T$ and $w \in X^*$ such that if $(f,g)_w$ is defined (it means that $a\overline{f}b = c\overline{g}$, where $a,b,c \in X^*,\ f \in S,\ g \in T$ and $deg(\overline{f}) > deg(c)$), we have $(f,g)_w \equiv 0 \ mod\ (S,T;w)$. In this case, we say that (S,T) is a Gröbner-Shirshov pair for the A-module ${}_AM ={}_A A/A(T+Id(S))$, where $A = k\langle X|S\rangle$.*

The following is the Kang-Lee's Composition-Diamond lemma for a left module.

Theorem 2.6 *([15],[19]) Let (S,T) be a pair of subsets of monic elements in $k\langle X\rangle$, let $A = k\langle X|S\rangle$ be the associative algebra defined by S, and let ${}_AM ={}_A A/A(T+Id(S))$ be the left module defined by (S,T). Suppose that (S,T) is a Gröbner-Shirshov pair for the A-module ${}_AM$ and $p \in k\langle X\rangle T + Id(S)$. Then $\overline{p} = a\overline{s}b$ or $\overline{p} = c\overline{t}$, where $a,b,c \in X^*,\ s \in S,\ t \in T$.*

In particular, Lemma 2.3 is a special case of Theorem 2.6 when $S = \varnothing$.

3 Composition-Diamond lemma for "double-free" modules

Let X,Y be sets and $mod_{k\langle X\rangle}\langle Y\rangle$ a free left $k\langle X\rangle$-module with the basis Y. Then $mod_{k\langle X\rangle}\langle Y\rangle = \oplus_{y\in Y} k\langle X\rangle y$ is called a "double-free" module.

We now define Gröbner-Shirshov basis in $mod_{k\langle X\rangle}\langle Y\rangle$.

Suppose that $<$ is a monomial order on X^*, $<$ a well order on Y and $X^*Y = \{uy|u \in X^*,\ y \in Y\}$. We define an order " $\prec$ " on X^*Y: for any $w_1 = u_1y_1, w_2 = u_2y_2 \in X^*Y$,

$$w_1 \prec w_2 \Leftrightarrow u_1 < u_2 \quad \text{or } u_1 = u_2, y_1 < y_2 \qquad (*)$$

It is clear that the order " $\prec$ " is left compatible in the sense of

$$w \prec w' \Rightarrow aw \prec aw' \ \text{ for any } a \in X^*.$$

Let $S \subset mod_{k\langle X\rangle}\langle Y\rangle$ with each $s \in S$ monic. We define the composition in S only the inclusion composition which means $w = \bar{f} = a\bar{g}$, where $f,g \in S$.

If $(f,g)_w = f - ag = \sum \alpha_i a_i s_i$, where $\alpha_i \in k,\ a_i \in X^*,\ s_i \in S$ and $a_i\overline{s_i} \prec w$, then this composition is called trivial modulo (S,w) and denote it by

$$(f,g)_w \equiv 0 \ mod(S,w).$$

Definition 3.1 *([11]) Let $S \subset mod_{k\langle X\rangle}\langle Y\rangle$ be a non-empty set with each $s \in S$ monic. Let the order "$\prec$" be as before. Then we call S a Gröbner-Shirshov basis in the module $mod_{k\langle X\rangle}\langle Y\rangle$ if all the compositions in S are trivial modulo S.*

The proof of the following lemma is essentially from [11]. For convenience, we give the detail.

Lemma 3.2 *([11], Composition-Diamond lemma for "double-free" modules) Let $S \subset mod_{k\langle X\rangle}\langle Y\rangle$ be a non-empty set with each $s \in S$ monic and " $\prec$ " the order on X^*Y as before. Then the following statements are equivalent:*

(1) S is a Gröbner-Shirshov basis in $mod_{k\langle X\rangle}\langle Y\rangle$.

(2) If $0 \neq f \in k\langle X\rangle S$, then $\overline{f} = a\overline{s}$ for some $a \in X^*,\ s \in S$.

(2') If $0 \neq f \in k\langle X\rangle S$, then $f = \sum \alpha_i a_i s_i$ with $a_1\overline{s}_1 \succ a_2\overline{s}_2 \succ \cdots$, where each $\alpha_i \in k,\ a_i \in X^*,\ s_i \in S$.

(3) $Red(S) = \{w \in X^*Y | w \neq a\overline{s},\ a \in X^*,\ s \in S\}$ is a k-linear basis for the factor $mod_{k\langle X\rangle}\langle Y|S\rangle = mod_{k\langle X\rangle}\langle Y\rangle / k\langle X\rangle S$.

Proof: (1) $\Rightarrow$ (2). Suppose that $0 \neq f \in k\langle X\rangle S$. Then $f = \sum \alpha_i a_i s_i$ for some $\alpha_i \in k,\ a_i \in X^*,\ s_i \in S$. Let $w_i = a_i\overline{s}_i$ and $w_1 = w_2 = \cdots = w_l \succ w_{l+1} \succeq \cdots$. We will prove that $\overline{f} = a\overline{s}$ for some $a \in X^*,\ s \in S$, by using induction on l and w_1. If $l = 1$, then the result is clear. If $l > 1$, then $a_1\overline{s}_1 = a_2\overline{s}_2$. Thus, we may assume that $a_1 = a_2 a,\ \overline{s}_2 = a\overline{s}_1$ for some $a \in X^*$. Now, by (1),

$$a_1 s_1 - a_2 s_2 = a_2 a s_1 - a_2 s_2 = a_2(a s_1 - s_2) = a_2 \sum \beta_j b_j u_j = \sum \beta_j a_2 b_j u_j,$$

where $\beta_j \in k,\ b_j \in X^*,\ u_j \in S$ and $b_j\overline{u}_j \prec \overline{s}_2$. Therefore, $a_2 b_j \overline{u}_j \prec w_1$. Now, by induction on l and w_1, we have the result.

It is clear that (2) is equivalent to (2′).

(2) $\Rightarrow$ (3). For any $0 \neq f \in mod_{k\langle X\rangle}\langle Y\rangle$, if $\bar{f} = u_1 \in Red(S)$, then $f = \beta_1 u_1 + \cdots$. If $\bar{f} \notin Red(S)$, then $f = \alpha_1 a_1 s_1 + \cdots$. It follows that we can express f as

$$f = \sum \alpha_i a_i s_i + \sum \beta_j u_j,$$

where $\alpha_i,\ \beta_j \in k,\ a_i \in X^*,\ s_i \in S$ and $u_j \in Red(S)$. Then $Red(S)$ generates the factor module. Moreover, assume that $0 \neq \sum \alpha_i a_i s_i = \sum \beta_j u_j$, where $a_i \in X^*,\ s_i \in S,\ u_j \in Red(S),\ a_1\overline{s}_1 \succ a_2\overline{s}_2 \succ \cdots$ and $u_1 \succ u_2 \succ \cdots$. Then $u_1 = a_1\overline{s}_1$, a contradiction. This shows that $Red(S)$ is a k-linear basis of the factor module.

(3) $\Rightarrow$ (1). For any $f,\ g \in S$, suppose that $w = \overline{f} = a\overline{g}$. Since $(f,g)_w \in k\langle X\rangle S$, we get, by (3), that $(f,g)_w = f - ag = \sum \alpha_i a_i s_i$, where $s_i \in S,\ a_i \in X^*$ and $a_i\overline{s_i} \preceq \overline{(f,g)_w} \prec w$. Then, S is a Gröbner-Shirshov basis in the module $mod_{k\langle X\rangle}\langle Y\rangle$. $\square$

Remark: Let us view $k\langle X\rangle$ as free left $k\langle X\rangle$-module with one generator I. Then $mod_{k\langle X\rangle}\langle I\rangle = k\langle X\rangle I =_{k\langle X\rangle} k\langle X\rangle$ is a cyclic $k\langle X\rangle$-module. If $S \subset k\langle X\rangle$, then $k\langle X\rangle S$ is a left ideal of $k\langle X\rangle$ (also a left $k\langle X\rangle$-submodule of $k\langle X\rangle I$). This implies that Lemma 2.3 is a special case of Lemma 3.2.

Let $S \subset k\langle X\rangle$ and $A = k\langle X|S\rangle$ an associative algebra. Then, for any left A-module ${}_AM$, we can view ${}_AM$ as a $k\langle X\rangle$-module in a natural way: for any $f \in k\langle X\rangle,\ m \in M$,

$$fm = (f + Id(S))m.$$

We note that ${}_AM$ is an epimorphic image of some free A-module. Then, we can assume that ${}_AM = mod_A\langle Y|T\rangle = mod_A\langle Y\rangle/AT$, where $T \subset mod_A\langle Y\rangle$ and $mod_A\langle Y\rangle$ a free left A-module with the basis Y. Let $T_1 = \{\sum f_i y_i \in mod_{k\langle X\rangle}\langle Y\rangle | \sum(f_i + Id(S))y_i \in T\}$ and $R = SX^*Y \cup T_1$. Then, by the following Lemma 3.3, we have, as $k\langle X\rangle$-modules, ${}_AM \cong mod_{k\langle X\rangle}\langle Y|R\rangle$.

Lemma 3.3 *([11]) Let the notations be as above. Then, as $k\langle X\rangle$-modules,*

$$\sigma : {}_AM \to mod_{k\langle X\rangle}\langle Y|R\rangle, \quad \sum(f_i + Id(S))(y_i + AT) \mapsto \sum f_i y_i + k\langle X\rangle R$$

is an isomorphism, where each $f_i \in k\langle X\rangle$.

Proof: For any $\sum(f_i + Id(S))(y_i + AT), \sum(g_i + Id(S))(y_i + AT) \in {}_AM$, we have

$$\begin{aligned}
& \sum(f_i + Id(S))(y_i + AT) = \sum(g_i + Id(S))(y_i + AT) \quad \text{in} \quad {}_AM \\
\Leftrightarrow \ & \sum(f_i - g_i)y_i \in AT \quad \text{in} \quad {}_AM \\
\Leftrightarrow \ & \sum(f_i - g_i)y_i \in k\langle X\rangle R \\
\Leftrightarrow \ & \sum f_i y_i + k\langle X\rangle R = \sum g_i y_i + k\langle X\rangle R.
\end{aligned}$$

Hence, σ is injective. It is easy to see that σ is a surjective mapping and a $k\langle X\rangle$-module homomorphism. □

By using Lemma 3.2 and Lemma 3.3, we know that if we want to find a k-linear basis for the module ${}_AM = mod_A\langle Y|T\rangle$, where $A = k\langle X|S\rangle$, we only need to find a Gröbner-Shirshov basis for the module $mod_{k\langle X\rangle}\langle Y|SX^*Y \cup T_1\rangle$, where $T_1 = \{\sum f_i y_i \in mod_{k\langle X\rangle}\langle Y\rangle | \sum(f_i + Id(S))y_i \in T\}$.

The following theorem gives some relationships between the Gröbner-Shirshov bases (pairs) in free associative algebras and "double-free" modules.

Theorem 3.4 *Let X, Y be well ordered sets, $<$ a monomial order on X^*, $\prec$ the order on X^*Y as in $(*)$. Let S, $T \subset k\langle X\rangle$ be monic sets. Then the following statements hold:*

(1) $S \subset k\langle X\rangle$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ with respect to the order $<$ if and only if $SX^*Y \subset mod_{k\langle X\rangle}\langle Y\rangle$ is a Gröbner-Shirshov basis in $mod_{k\langle X\rangle}\langle Y\rangle$ with respect to the order $\prec$.

(2) Let us view $k\langle X\rangle$ as free $k\langle X\rangle$-module with one generator I. Then, (S,T) is a Gröbner-Shirshov pair for the A-module $M = A/A(T + Id(S))$, where $A = k\langle X|S\rangle$, if and only if S is a Gröbner-Shirshov basis in the algebra $k\langle X\rangle$ with respect to the order $<$ and $(SX^* \cup T)I$ a Gröbner-Shirshov basis in the free module $mod_{k\langle X\rangle}\langle I\rangle$ with respect to the order $\prec$.

Proof: (1) Suppose that S is a Gröbner-Shirshov basis in $k\langle X\rangle$. We shall prove that all the compositions in SX^*Y are trivial modulo SX^*Y. For any f, $g \in SX^*Y$, let

$f = s_1a_1y,\ g = s_2a_2y,\ s_1, s_2 \in S,\ a_1, a_2 \in X^*,\ y \in Y$ and $w = \overline{f} = a\overline{g}$. Then $\overline{s}_1a_1 = a\overline{s}_2a_2$. Since S is a Gröbner-Shirshov basis in $k\langle X\rangle$, we have

$$(f, g)_w = f - ag = s_1a_1y - as_2a_2y = (s_1a_1 - as_2a_2)y = \sum(\alpha_iu_ir_iv_i)y,$$

where $u_i,\ v_i \in X^*,\ r_i \in S$ and $u_i\overline{r}_iv_iy \prec w$. So each composition is trivial modulo SX^*Y and hence, SX^*Y is a Gröbner-Shirshov basis in $mod_{k\langle X\rangle}\langle Y\rangle$.

Conversely, assume that SX^*Y is a Gröbner-Shirshov basis in the module $mod_{k\langle X\rangle}\langle Y\rangle$. For any $f,\ g \in S$ and $w = \overline{f}a = b\overline{g}$, we have $w_1 = \overline{fay} = b\overline{gy}$ and

$$(fay, bgy)_{w_1} = (fa - bg)y = \sum \alpha_i(a_ir_i)y,$$

where $\alpha_i \in k,\ r_i = s_ib_i,\ a_i, b_i \in X^*,\ s_i \in S$ and $a_i\overline{r}_iy \prec w_1$. Then,

$$(f, g)_w = fa - bg = \sum \alpha_ia_is_ib_i$$

with $a_i\overline{s}_ib_i < w$. This shows that, each composition of intersection in S is trivial modulo S. Similarly, each composition of inclusion in S is trivial modulo S. Therefore, S is a Gröbner-Shirshov basis in $k\langle X\rangle$.

(2) The results follow from the Definitions 2.5 and 3.1 directly. $\square$

Remark: By Theorem 3.4, it is clear that the Theorem 2.6 follows from Lemma 3.2.

4 Highest weight modules over sl_2

In this section, as an application of Lemma 3.2, we re-prove that the highest weight module over sl_2 is irreducible (see [12]) and we show that any finite dimensional irreducible sl_2-module has the presentation $(**)$ as below.

Let $X = \{x,\ y,\ h\}$ and $sl_2 = Lie(X|S)$ a Lie algebra over a field k with $chk = 0$, where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } S = \{[hx]-2x,\ [hy]+2y,\ [xy]-h\}.$$

Then the universal enveloping algebra of sl_2 is $\mathcal{U}(sl_2) = k\langle X|\ S^{(-)}\rangle$. Define the deg-lex order on X^* with $x > h > y$. Then S is a Gröbner-Shirshov basis in free Lie algebra $Lie(X)$ since $S^{(-)}$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ (see Lemma 1.1). Let

$${}_{sl_2}V(\lambda) = mod_{sl_2}\langle v_0|xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0\rangle$$

be a highest weight module generated by v_0 with the highest weight λ. We can rewrite it as

$$\begin{aligned} {}_{sl_2}V(\lambda) &= mod_{\mathcal{U}(sl_2)}\langle v_0|\ xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0\rangle \\ &= mod_{k\langle X\rangle}\langle v_0|\ xv_0 = 0,\ hv_0 = \lambda v_0,\ y^{m+1}v_0 = 0,\ S^{(-)}X^*v_0 = 0\rangle \end{aligned}$$

Let $S_1 = \{xv_0,\ hv_0 - \lambda v_0,\ y^{m+1}v_0\} \cup S^{(-)}X^*v_0$. It is easy to see that all compositions in S_1 are trivial modulo S_1. Thus, S_1 is a Gröbner-Shirshov basis for this module with

respect to the order as (*) in §3, and by Lemma 3.2, $Red(S_1) = \{y^i v_0 | 0 \leq i \leq m\}$ is a k-linear basis for module ${}_{sl_2}V(\lambda)$, and so $dim(V(\lambda)) = m + 1$.

Let $y^{(i)} = \frac{1}{i!}y^i$, $v_i = \frac{1}{i!}y^i v_0$ and $v_{-1} = 0$. Then v_i $(0 \leq i \leq m)$ is also a linear basis of $V(\lambda)$. Now, using ELW of the relations in S_1 on the left parts, we have the following equalities (see also [12], p.32):

Lemma 4.1

$$\begin{aligned} hv_i &= (\lambda - 2i)v_i \\ yv_i &= (i+1)v_{i+1} \\ xv_i &= (\lambda - i + 1)v_{i-1} \ (0 \leq i) \qquad \square \end{aligned}$$

Since $v_{m+1} = 0$ and $chk = 0$, $0 = xv_{m+1} = (\lambda - m)v_m$ and therefore, $\lambda = m$.

Lemma 4.2 *$V(\lambda)$ is irreducible.*

Proof: Let $0 \neq V_1 \leq V(\lambda)$ be a submodule. Since $V_1 \neq 0$, there exist $0 \neq a_i v_i + a_{i+1}v_{i+1} + \cdots + a_m v_m$, where i is the least number such that $a_i \neq 0$. Applying y to it $m - i$ times, we get $a_i(i+1)(i+2)\cdots m v_m \in V_1$. So, $v_m \in V_1$. Applying x to v_m, we get $v_i \in V_1$ $(0 \leq i < m)$ and $V_1 = V(\lambda)$. $\square$

For any finite dimensional irreducible sl_2-module V, choosing a maximal vector $v_0 \in V$ and $v_i = \frac{1}{i!}y^i v_0$, we have the formulas as in Lemma 4.1. We can suppose that $dimV = m$. Thus, $v_m \neq 0$, $v_{m+1} = 0$ and hence, V can be represented as

$${}_{sl_2}V = mod_{sl_2}\langle v_0 | xv_0 = 0, \ hv_0 = \lambda v_0, \ y^{m+1}v_0 = 0\rangle \qquad (**)$$

which means that any finite dimensional irreducible sl_2-module has the above form.

5 Verma module over Kac-Moody algebras

In this section, we give the definitions of Kac-Moody algebra $\mathcal{G}(A)$ and Verma module over $\mathcal{G}(A)$. Then, by using Lemma 3.2, we find a Gröbner-Shirshov basis for this Verma module.

Let $A = (a_{ij})$ be an (integral) symmetrizable n-by-n Cartan matrix over $\mathbb{C}$, where $\mathbb{C}$ is the complex field. It means that $a_{ii} = 2$, $a_{ij} \leq 0$ $(i \neq j)$, and there exists a diagonal matrix D with diagonal entries d_i nonzero integers such that product DA is symmetric. Let $\mathcal{G}(A) = Lie(Z|S)$ be a Lie algebra, where $Z = \{x_i, \ y_i, \ h | 1 \leq i \leq n, \ h \in H\}$, S consists of the following relations (see [13], p.159):

(5.1) $[x_i, y_j] = \delta_{ij}\alpha_i^\vee \quad (i, j = 1, \cdots, n)$,

(5.2) $[h, h'] = 0 \qquad (h, h' \in H)$,

(5.3) $[h, x_i] = \langle\alpha_i, h\rangle x_i, \ [h, y_i] = -\langle\alpha_i, h\rangle y_i, \ (i = 1, \cdots, n; h \in H)$,

(5.4) $(adx_i)^{1-a_{ij}}x_j = 0, \ (ady_i)^{1-a_{ij}}y_j = 0 \qquad (i \neq j)$,

where ad is the derivation, H a complex vector space, $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset H^\star$ (the dual space of H) and $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\} \subset H$ indexed subsets in $H^\star$ and H, respectively, satisfying the following conditions (see [13], p.1):

(a) both sets Π and $\Pi^\vee$ are linearly independent,

(b) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \qquad (i,\ j = 1, \cdots, n)$,

(c) $n - l = dimH - n \qquad rank(A) = l$.

Then we call this Lie algebra $\mathcal{G}(A)$ Kac-Moody algebra. Let $\mathfrak{N}_+$ ($\mathfrak{N}_-$) be the subalgebra of $\mathcal{G}(A)$ generated by x_i (y_i), $(0 \le i \le n)$. Then $\mathcal{G}(A) = \mathfrak{N}_- \oplus H \oplus \mathfrak{N}_+$ and $\mathcal{U}(\mathcal{G}(A)) = \mathcal{U}(\mathfrak{N}_+) \otimes k[H] \otimes \mathcal{U}(\mathfrak{N}_-)$ is the universal enveloping algebra of $\mathcal{G}(A)$, where $\mathcal{U}(\mathfrak{N}_+)$ ($\mathcal{U}(\mathfrak{N}_-)$) is the universal enveloping algebra of $\mathfrak{N}_+$ ($\mathfrak{N}_-$).

Let $\{h_j | 1 \le j \le 2n - l\}$ be a basis of H. We order the set $X = \{x_i,\ h_j,\ y_m | 1 \le i, m \le n,\ 1 \le j \le 2n - l\}$ by $x_i > x_j,\ h_i > h_j,\ y_i > y_j$, if $i > j$, and $x_i > h_j > y_m$ for all $i,\ j,\ m$. Then we define the deg-lex order on X^*.

By [8], we can get a Gröbner-Shirshov basis T for $\mathcal{U}(\mathcal{G}(A))$, where T consists of the following relations:

(5.5) $h_i h_j - h_j h_i,\ x_j h_i - h_i x_j + d_i a_{ij} x_i,\ h_i y_j - y_j h_i + d_i a_{ij} y_j,$

(5.6) $x_i y_j - y_j x_i - \delta_{ij} h_i,$

(5.7) $\{\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix} x_i^{1-a_{ij}-\nu} x_j x_i^\nu\}^c \qquad (i \neq j),$

(5.8) $\{\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix} y_i^{1-a_{ij}-\nu} y_j y_i^\nu\}^c \qquad (i \neq j).$

where S^c means a Gröbner-Shirshov basis that contains S.

Definition 5.1 ([13]) A $\mathcal{G}(A)$-module V is called a highest weight module with highest weight $\Lambda \in H^\star$ if there exist a non-zero vector $v \in V$, such that

$$\mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v,\ h \in H$$

and $\mathcal{U}(\mathcal{G}(A))(v) = V$.

A Verma module $M(\Lambda)$ with highest weight Λ has the following presentation:

$${}_{\mathcal{G}(A)}M(\Lambda) = mod_{\mathcal{G}(A)}\langle v | \mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v, \quad h \in H\rangle.$$

Corollary 5.2 *$R = \{TX^*(v),\ \mathfrak{N}_+(v),\ h(v) - \Lambda(h)v\}$ is a Gröbner-Shirshov basis for the Verma module ${}_{\mathcal{G}(A)}M(\Lambda)$.*

Proof: Since

$$\begin{aligned} {}_{\mathcal{G}(A)}M(\Lambda) &= mod_{\mathcal{U}(\mathcal{G}(A))}\langle v | \mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v, \quad h \in H\rangle \\ &= mod_{k\langle Z\rangle}\langle v | TX^*(v) = 0,\ \mathfrak{N}_+(v) = 0,\ h(v) = \Lambda(h)v,\ h \in H\rangle, \end{aligned}$$

it is easily to check that all compositions in R are trivial. So, R is Gröbner-Shirshov basis for the Verma module. $\square$

Remark: In the book [12], the author only consider the semisimple Lie algebras and call this highest weight module to be standard cyclic module.

6 Verma module over the coefficient algebra of a free Lie conformal algebra

In this section, by using Lemma 3.2, we find a basis of Verma module over Lie algebras of coefficients of free conformal algebras.

Let $\mathcal{B}$ be a set of symbols. Let the locality function $N : \mathcal{B}\times\mathcal{B}\rightarrow\mathbb{Z}_+$ be a constant, i.e. $N(a,b)\equiv N$ for any $a,\ b\in\mathcal{B}$. Let $X=\{b(n)|\ b\in\mathcal{B},\ n\in\mathbb{Z}\}$ and $L=Lie(X|S)$ be a Lie algebra generated by X with the relation S, where

$$S=\{\sum_s(-1)^s\left(\begin{array}{c}n\\s\end{array}\right)[b(n-s)a(m+s)]=0|\ a,\ b\in\mathcal{B},\ m,n\in\mathbb{Z}\}.$$

For any $b\in\mathcal{B}$, let $\widetilde{b}=\sum\limits_n b(n)z^{-n-1}\in L[[z,z^{-1}]]$. It is well-known that they generate a free Lie conformal algebra C with data $(\mathcal{B},N)$ (see [25]). Moreover, the coefficient algebra of C is just L.

Let $\mathcal{B}$ be a linearly ordered set. Define an order on X in the following way:

$$a(m)<b(n)\Leftrightarrow m<n\ \text{ or }(m=n\text{ and }a<b).$$

We use the deg-lex order on X^*. Then, it is clear that the leading term of each polynomial in S is $b(n)a(m)$ such that

$$n-m>N\text{ or }(n-m=N\text{ and }(b>a\text{ or }(b=a\text{ and }N\text{ is odd})).$$

The following lemma is from [25].

Lemma 6.1 *S is a Gröbner-Shirshov basis in $Lie(X)$.*

Corollary 6.2 *Let $\mathcal{U}=\mathcal{U}(L)$ be the universal enveloping algebra of L. Then a k-basis of $\mathcal{U}$ consists of monomials*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k),\ a_i\in\mathcal{B},\ n_i\in\mathbb{Z}$$

such that for any $1\leq i<k$,

$$n_i-n_{i+1}\leq\left\{\begin{array}{ll}N-1 & \textit{if } a_i>a_{i+1}\textit{ or }(a_i=a_{i+1}\ \textit{ and }\ N\textit{ is odd})\\ N & \textit{otherwise}\end{array}\right.\qquad(***)$$

Proof: Viewing $\mathcal{U}$ as a $k\langle X\rangle$-module, we have

$$_{\mathcal{U}}\mathcal{U}=mod_{\mathcal{U}}\langle X|\ S^{(-)}\rangle=mod_{k\langle X\rangle}\langle I|\ S^{(-)}X^*I\rangle.$$

Since S is a Gröbner-Shirshov basis in $Lie(X)$, $S^{(-)}$ is a Gröbner-Shirshov basis in $k\langle X\rangle$ by Lemma 1.1. Therefore, by Theorem 3.4, $S^{(-)}X^*I$ is a Gröbner-Shirshov basis in the free module $mod_{k\langle X\rangle}\langle I\rangle$. Now, the result follows from Lemma 3.2. $\square$

Definition 6.3 ([13] [14])

(a) An L-module M is called restricted, if for any $a \in C,\ v \in M$ there is some integer T such that for any $n \geq T$ one has $a(n)v = 0$.

(b) An L-module M is called highest weight module if it is generated over L by a single element $m \in M$ such that $L_+ m = 0$, where L_+ is the subspace of L generated by $\{a(n)|a \in C, n \geq 0\}$. In this case m is called the highest weight vector.

Now we build a universal highest weight module V over L, which is often referred to as Verma module. Let kI_v be a 1-dimensional trivial L_+-module generated by I_v, i.e., $a(n)I_v = 0$ for all $a \in \mathcal{B},\ n \geq 0$. Clearly,

$$V = Ind_{L_+}^{L} kI_v = \mathcal{U}(L) \otimes_{\mathcal{U}(L_+)} kI_v \cong \mathcal{U}(L)/\mathcal{U}(L)L_+.$$

Then V has a structure highest weight module over L with the action given by the multiplication on $\mathcal{U}(L)/\mathcal{U}(L)L_+$ and the highest weight vector $I \in \mathcal{U}(L)$. Also $V = \mathcal{U}(L)/\mathcal{U}(L)L_+$ is the universal enveloping vertex algebra of C and the embedding $\varphi : C \to V$ is given by $a \mapsto a(-1)I$ (see also [25]).

Theorem 6.4 *Let the notions be as above. Then a k-basis of V consists of elements*

$$a_1(n_1)a_2(n_2)\cdots a_k(n_k),\ a_i \in \mathcal{B},\ n_i \in \mathbb{Z}$$

such that the condition $(***)$ holds and $n_k < 0$.

Proof: Clearly, as $k\langle X\rangle$-modules,

$${}_{\mathcal{U}}V =_{\mathcal{U}} (\mathcal{U}(L)/\mathcal{U}(L)L_+) = mod_{k\langle X\rangle}\langle I|\ S^{(-)}X^*I,\ a(n)I,\ n \geq 0\rangle =_{k\langle X\rangle} \langle I|\ S'\rangle,$$

where $S' = \{S^{(-)}X^*I,\ a(n)I,\ n \geq 0\}$. In order to prove that S' is a Gröbner-Shirshov basis, we only need to check $w = b(n)a(m)I$, where $m \geq 0$. Let

$$f = \sum_s (-1)^s \binom{n}{s} (b(n-s)a(m+s) - a(m+s)b(n-s))I \quad and \quad g = a(m)I.$$

Then $(f,g)_w = f - b(n)a(m)I \equiv 0\ mod(S', w)$ since $n - m \geq N,\ m+s \geq 0,\ n-s \geq 0,\ 0 \leq s \leq N$. It follows that S' is a Gröbner-Shirshov basis. Now, the result follows from Lemma 3.2. $\square$

7 Universal enveloping module for a Sabinin algebra

In this section, we deal with the universal enveloping module for a Sabinin algebra.

Definition 7.1 ([21]) A vector space V is called a Sabinin algebra if it is endowed with multilinear operation $\langle ; \rangle$: for any $x_1, x_2, \cdots, x_m, y, z \in V$ and any $m \geq 0$,

$$\langle x_1, x_2, \cdots, x_m; y, z\rangle$$

satisfying the identities

$$\begin{aligned}
&\langle x_1, x_2, \cdots, x_m; y, z\rangle = -\langle x_1, x_2, \cdots, x_m; z, y\rangle,\\
&\langle x_1, x_2, \cdots, x_r, a, b, x_{r+1}, \cdots, x_m; y, z\rangle - \langle x_1, x_2, \cdots, x_r, b, a, x_{r+1}, \cdots, x_m; y, z\rangle\\
&\qquad + \sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1}, \cdots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \cdots, x_{\alpha_r}; a, b\rangle, \cdots, x_m; y, z\rangle = 0,\\
&\sigma_{x,y,z}(\langle x_1, x_2, \cdots, x_r, x; y, z\rangle + \sum_{k=0}^{r}\sum_{\alpha}\langle x_{\alpha_1}, \cdots, x_{\alpha_k}; \langle x_{\alpha_{k+1}}, \cdots, x_{\alpha_r}; y, z\rangle, x\rangle) = 0,
\end{aligned}$$

where α runs the set of all bijections of the type $\alpha:\ \{1, 2, \cdots, r\} \to \{1, 2, \cdots, r\},\ i \mapsto \alpha_i,\ \alpha_1 < \alpha_2 < \cdots < \alpha_k,\ \alpha_{k+1} < \cdots < \alpha_r,\ r \geq 0,\ \sigma_{x,y,z}$ denotes the cyclic sum by x, y, z.

Let $X = \{a_i | i \in \Lambda\}$ be a totally ordered basis of V. We define the deg-lex order on X^*. Let $\Delta:\ V \to V \otimes V$ be a linear map which satisfies: $\Delta(a_i) = 1 \otimes a_i + a_i \otimes 1,\ (Id \otimes \Delta)\Delta = (\Delta \otimes Id)\Delta$ (coassociative) and if $\tau\Delta = \Delta$ then $\tau(x \otimes y) = y \otimes x$ (cocommutative). It is customary to write $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Let $T(V)$ be the tensor algebra over V endowed with its natural structure of cocommutative Hopf algebra, that is, $V \subseteq Prim(T(V))$ (the primitive element of $T(V)$). Let $\langle ; \rangle : T(V) \bigotimes V \bigotimes V \to V$ be a map. Then we may write the definition of a Sabinin algebra very shortly as

$$\langle x; a, b\rangle = -\langle x; b, a\rangle,$$

$$\langle x[a, b]y; c, e\rangle + \sum \langle x_{(1)} \langle x_{(2)}; a, b\rangle y; c, e\rangle = 0,$$

$$\sigma_{a,b,c}(\langle xc; a, b\rangle + \sum \langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle) = 0,$$

where $[a, b] = ab - ba$.

Definition 7.2 ([21]) Let $(V, \langle ; \rangle)$ be a Sabinin algebra. Then

$$\widetilde{S}(V) = T(V)/span\langle xaby - xbay + \sum x_{(1)} \langle x_{(2)}; a, b\rangle y | x, y \in T(V), a, b \in V\rangle$$

is called the universal enveloping module for V.

Since $T(V) \simeq k\langle X\rangle$ as k-algebras, we can view $\widetilde{S}(V)$ as a right $k\langle X\rangle$-module:

$$\widetilde{S}(V) = mod\langle X | I\rangle_{k\langle X\rangle},$$

where $I = \{xab - xba + \sum x_{(1)} \langle x_{(2)}; a, b\rangle | x \in X^*, a > b,\ a, b \in X\}$.

Then we have the following theorem.

Theorem 7.3 *Let I be as above. Then I is the Gröbner-Shirshov basis in $mod\langle X\rangle_{k\langle X\rangle}$.*

Proof: There are two kinds of compositions: $w_1 = xabc$ $(a > b > c)$ and $w_2 = ucdvab$ $(c > d, a > b)$. Denote by

$$\begin{aligned}
f_1 &= xabc - xacb + \sum (xa)_{(1)}\langle (xa)_{(2)}; b, c\rangle, \\
f_2 &= xab - xba + \sum x_{(1)}\langle x_{(2)}; a, b\rangle, \\
f_3 &= ucdvab - ucdvba + \sum (ucdv)_{(1)}\langle (ucdv)_{(2)}; a, b\rangle, \\
f_4 &= ucd - udc + \sum u_{(1)}\langle u_{(2)}; c, d\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
(f_1, f_2)_{w_1} &= xabc - xacb + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle \\
&\quad -xabc + xbac - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c \\
&\equiv -xcab + \sum x_{(1)}\langle x_{(2)}; a, c\rangle b + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle \\
&\quad +xbca - \sum x_{(1)}b\langle x_{(2)}; a, c\rangle - \sum x_{(1)}\langle x_{(2)}b; a, c\rangle - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c \\
&\equiv \sum x_{(1)}c\langle x_{(2)}; a, b\rangle + \sum x_{(1)}\langle x_{(2)}c; a, b\rangle + \sum x_{(1)}\langle x_{(2)}; a, c\rangle b \\
&\quad + \sum x_{(1)}a\langle x_{(2)}; b, c\rangle + \sum x_{(1)}\langle x_{(2)}a; b, c\rangle - \sum x_{(1)}\langle x_{(2)}; b, c\rangle a \\
&\quad - \sum x_{(1)}b\langle x_{(2)}; a, c\rangle - \sum x_{(1)}\langle x_{(2)}b; a, c\rangle - \sum x_{(1)}\langle x_{(2)}; a, b\rangle c \\
&\equiv \sum x_{(1)}\langle x_{(2)}a; b, c\rangle + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; b, c\rangle, a\rangle + \sum x_{(1)}\langle x_{(2)}b; c, a\rangle \\
&\quad + \sum x_{(1)}\langle x_{(2)}c; a, b\rangle + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; c, a\rangle, b\rangle \\
&\quad + \sum x_{(1)}\langle x_{(2)}; \langle x_{(3)}; a, b\rangle, c\rangle \\
&\equiv 0
\end{aligned}$$

since $\sigma_{a,b,c}(\langle xc; a, b\rangle + \sum \langle x_{(1)}; \langle x_{(2)}; a, b\rangle, c\rangle) = 0$.

$$
\begin{aligned}
(f_3,f_4)_{w_2} &= ucdvab-ucdvba+\sum u_{(1)}v_{(1)}\langle u_{(2)}cdv_{(2)};a,b\rangle+\sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)};a,b\rangle+\sum u_{(1)}cdv_{(1)}\langle u_{(2)}v_{(2)};a,b\rangle\\
&\quad -ucdvab+udcvab-\sum u_{(1)}\langle u_{(2)};c,d\rangle vab\\
&\equiv -udcvba+\sum u_{(1)}\langle u_{(2)};c,d\rangle vba+\sum u_{(1)}v_{(1)}\langle u_{(2)}cdv_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)};a,b\rangle+\sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}cdv_{(1)}\langle u_{(2)}v_{(2)};a,b\rangle+udcvba-\sum u_{(1)}v_{(1)}\langle u_{(2)}dcv_{(2)};a,b\rangle\\
&\quad -\sum u_{(1)}cv_{(1)}\langle u_{(2)}dv_{(2)};a,b\rangle-\sum u_{(1)}dv_{(1)}\langle u_{(2)}cv_{(2)};a,b\rangle\\
&\quad -\sum u_{(1)}dcv_{(1)}\langle u_{(2)}v_{(2)};a,b\rangle+\sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)};c,d\rangle v_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}\langle u_{(2)};c,d\rangle v_{(1)}\langle u_{(3)}v_{(2)};a,b\rangle-\sum u_{(1)}\langle u_{(2)};c,d\rangle vba\\
&\equiv \sum u_{(1)}v_{(1)}\langle u_{(2)}[c,d]v_{(2)};a,b\rangle+\sum u_{(1)}[c,d]v_{(1)}\langle u_{(2)}v_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)};c,d\rangle v_{(2)};a,b\rangle+\sum u_{(1)}\langle u_{(2)};c,d\rangle v_{(1)}\langle u_{(3)}v_{(2)};a,b\rangle\\
&\equiv \sum(u_{(1)}[c,d]+u_{(1)}\langle u_{(2)};c,d\rangle)v_{(1)}\langle u_{(3)}v_{(2)};a,b\rangle\\
&\quad +\sum u_{(1)}v_{(1)}\langle u_{(2)}[c,d]v_{(2)};a,b\rangle+\sum u_{(1)}v_{(1)}\langle u_{(2)}\langle u_{(3)};c,d\rangle v_{(2)};a,b\rangle\\
&\equiv 0
\end{aligned}
$$

since $\langle x[a,b]y;c,e\rangle+\sum\langle x_{(1)}\langle x_{(2)};a,b\rangle y;c,e\rangle=0$.

Hence, I is a Gröbner-Shirshov basis in $mod\langle X\rangle_{k\langle X\rangle}$. $\square$

Remark: From the above proof, we know that for $\widetilde{S}(V)=mod\langle X|I\rangle_{k\langle X\rangle}$, the minimal Gröbner-Shirshov basis is
$G=\{xab-xba+\sum x_{(1)}\langle x_{(2)};a,b\rangle|x=a_{i_1}\cdots a_{i_n}(i_1\le\cdots\le i_n,\ n\ge 0),a>b,\ a,b\in X\}$.

Now, by Lemma 3.2 and Theorem 7.3, we can easily get the following theorem.

Theorem 7.4 ([21], Poincare-Birkhoff-Witt) Let $\{a_i|i\in\Lambda\}$ be a totally ordered basis of V. Then $\{a_{i_1}\cdots a_{i_n}|i_1\le i_2\le\cdots\le i_n,\ n\ge 0\}$ is a basis of $\widetilde{S}(V)$.

Acknowledgement: The authors would like to express their deepest gratitude to Professor L. A. Bokut for his kind guidance, useful discussions and enthusiastic encouragement. We also thank Professor J. M. Perez-Izquierdo for some remarks.

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